数理解析研究所講究録 1258 巻 2002 年 85-93

Backward global solutions characterizing annihilation dynamics of travelling fronts

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Abstract We consider a reaction-diffusion equation $u_t = u_{xx} + f(u)$, where f has exactly three zeros 0, α and 1 $(0 < \alpha < 1)$, $f_u(0) < 0$, $f_u(1) < 0$ and $\int_0^1 f(u)du \geq 0$. Then, the equation has a travelling wave solution $u(x,t) = \phi(x - ct)$ with $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. Known results suggest that for an initial state $u_0(x)$ with $\underline{\lim}_{x\to\pm\infty} u_0(x) > \alpha$ having two interfaces at a large distance, $u(x,t)$ approaches a pair of travelling wave solutions $\phi(x - p_1(t)) + \phi(-x + p_2(t))$ for a long time, and then the travelling fronts eventually disappear by colliding with each other. While our results establish this process, they show that *there is a (backward) global solution* $\psi(x,t)$ and that *the annihilation process is approximated by a solution* $\psi(x - x_0, t - t_0)$.

Keywords: bistable reaction-diffusion equation, entire solution, travelling wave, collision, collapse, invariant manifold.

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preprint: RIMS-1311
http://www.kurims.kyoto-u.ac.jp/~kenkyubu/paper/all.html

1 Introduction

In this paper, we consider the scalar bistable reaction-diffusion equation

$$(1.1) \qquad \begin{cases} u_t = u_{xx} + f(u), & t > 0, \quad x \in \mathbf{R}, \\ u(0) = u_0 \in BU(\mathbf{R}), \end{cases}$$

where $BU(\mathbf{R})$ is the space of bounded uniformly continuous functions from $\mathbf{R}$ to $\mathbf{R}$ with the supremum norm, and the reaction term f satisfies the following conditions:

1 $f \in C^2(\mathbf{R})$,

2 f has exactly three zeros 0, α and 1 $(0 < \alpha < 1)$,

3 $f_u(0) < 0,\ f_u(1) < 0$,

4 $\int_0^1 f(u)du \geq 0$.

It is known (e.g. [4, Section 4.4]) that the reaction-diffusion equation (1.1) has a unique (except for translation) travelling wave solution $u(x,t) = \phi(x - ct)$, where (ϕ, c) satisfies

$$(1.2) \qquad \phi''(z) + c\phi'(z) + f(\phi(z)) = 0$$

with $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. Then $c \leq 0$ holds from $\int_0^1 f(u)du \geq 0$. We normalize the definition of ϕ by requiring $\phi(0) = 1/2$.

This solution is linearly stable except for neutral translational perturbations. Specifically, the following is known (e.g. [10, Section 5.4]).

Theorem A *(1) The operator* $-(\frac{\partial^2}{\partial z^2} + c\frac{\partial}{\partial z} + f_u(\phi(z))) : BU(\mathbf{R}) \to BU(\mathbf{R})$ *is a sectorial one with a simple eigenvalue* 0*. The remainder of the spectrum has real part greater than some positive constant.*

(2) There exist δ*,* C *and* $\gamma > 0$ *such that for any* $u_0 \in BU(\mathbf{R})$ *with* $\|u_0(x) - \phi(x)\|_{C^0} \leq \delta$*, there exists* $x_0 \in \mathbf{R}$ *satisfying*

$$\|u(x,t) - \phi(x - x_0 - ct)\|_{C^0} \leq Ce^{-\gamma t}\|u_0(x) - \phi(x)\|_{C^0}$$

for all $t \geq 0$.

Moreover, Fife and McLeod [6] showed the following theorem, which gives a global stability result for the travelling wave solution $\phi(x-ct)$.

Theorem B *If* $\overline{\lim}_{x\to-\infty}u_0(x)<\alpha$ *and* $\underline{\lim}_{x\to+\infty}u_0(x)>\alpha$ *hold, then*

$$\inf_{x_0\in\mathbf{R}}\|u(x,t)-\phi(x-x_0)\|_{C^0}\to 0 \quad as \quad t\to+\infty$$

holds.

Also, Fife and McLeod [6] showed the following, which means that the pair of the travelling wave solutions going to $x=\pm\infty$ has strong attractivity.

Theorem C *Suppose that* $c<0$, $\overline{\lim}_{x\to\pm\infty}u_0(x)<\alpha$, $u_0(x)\geq\eta$ $(|x|<L)$ *for some* $\eta>\alpha$ *and* $u_0(x)\geq\zeta$ $(|x|<\infty)$ *for some* $\zeta>-\infty$ *hold. If* L *is large enough depending on* η *and* ζ*, then* $u(x,t)$ *approaches (uniformly in* x *and exponentially in* t*) a pair of diverging travelling wave solutions*

$$\phi(x-x_1-ct)+\phi(-x-x_2-ct)-1.$$

On the other hand, when $\underline{\lim}_{x\to\pm\infty}u_0(x)>\alpha$ holds, the following is known (e.g. [5]).

Proposition D *If* $\underline{\lim}_{x\to\pm\infty}u_0(x)>\alpha$ *holds, then* $\lim_{t\to+\infty}\|u(x,t)-1\|_{C^0}=0$ *holds.*

For an initial state $u_0(x)$ with $\underline{\lim}_{x\to\pm\infty}u_0(x)>\alpha$ having two interfaces at a large distance, Theorems A, B and C suggest that $u(x,t)$ approaches a pair of travelling wave solutions

$$\phi(x-p_1(t))+\phi(-x+p_2(t))$$

for a long time. Then, Proposition D suggests that the travelling fronts eventually disappear by colliding with each other. While our main results (Theorem 1.1 and Corollary 1.4) establish this process, they show that *there is a (backward) global solution* $\psi(x,t)$ and that *the annihilation process is approximated by a solution* $\psi(x-x_0,t-t_0)$.[1]

Theorem 1.1 *There exists a solution* $\psi\in C(\mathbf{R},BU(\mathbf{R}))$ *of* $u_t=u_{xx}+f(u)$ *satisfying* $\lim_{t\to+\infty}\|\psi(t)-1\|_{C^0(\mathbf{R})}=0$, $\psi(-x,t)=\psi(x,t)$ *and the following.*

[1] For mathematical studies on motion and collapse of fronts in (1.1) from other aspects, we can refer to, e.g., [1], [2], [3], [7], [8], [9], [11] and [12].

(1) There exists $p \in C^1(\mathbf{R})$ *such that*

$$p(-\infty) = +\infty, \ \dot{p}(-\infty) = c$$

and

$$\lim_{t \to -\infty} \|\psi(x,t) - (\phi(x - p(t)) + \phi(-x - p(t)))\|_{C^0(\mathbf{R})} = 0$$

hold.

(2) There exist $\delta > 0$, $C > 0$ *and* $\gamma > 0$ *such that for any* $t_0 \in \mathbf{R}$ *and* $u_0 \in BU(\mathbf{R})$ *satisfying* $\|u_0 - \psi(t_0)\|_{C^0(\mathbf{R})} \leq \delta$, *there exist* x_0, $t_0' \in \mathbf{R}$ *and a solution* $u \in C([0, +\infty), BU(\mathbf{R}))$ *of* $u_t = u_{xx} + f(u)$ *with* $u(0) = u_0$ *such that*

$$\|u(x,t) - \psi(x - x_0, t - t_0')\|_{C^0(\mathbf{R})} \leq Ce^{-\gamma t}\|u_0(x) - \psi(x, t_0)\|_{C^0(\mathbf{R})}$$

holds for all $t \geq 0$.

Theorem 1.1 leads to the following. This is a uniqueness result for the global solution $\psi(x,t)$.

Corollary 1.2 *For any* $T \in [-\infty, +\infty)$ *and solution* $\bar{\psi} \in C((T, +\infty), BU(\mathbf{R}))$ *of* $u_t = u_{xx} + f(u)$, *if there exist* $\{p_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty} \subset \mathbf{R}$ *and* $\{T_n\}_{n=1}^{\infty} \subset (T, +\infty)$ *such that*

$$\lim_{n \to \infty} (p_n - q_n) = +\infty$$

and

$$\lim_{n \to \infty} \|\bar{\psi}(x, T_n) - (\phi(x - p_n) + \phi(-x + q_n))\|_{C^0(\mathbf{R})} = 0 \tag{1.3}$$

hold, then $T = -\infty$ *holds and there exist* x_0 *and* $t_0 \in \mathbf{R}$ *satisfying*

$$\psi(x,t) = \bar{\psi}(x + x_0, t + t_0).$$

Proof. By Theorem 1.1 (1), there exists $\{t_n'\}_{n=1}^{\infty} \subset \mathbf{R}$ with $\lim_{n \to \infty} t_n' = -\infty$ such that

$$\lim_{n \to \infty} \|\psi(x, t_n') - (\phi(x - \frac{p_n - q_n}{2}) + \phi(-x - \frac{p_n - q_n}{2}))\|_{C^0(\mathbf{R})} = 0$$

holds. Hence, from (1.3),

$$\lim_{n \to \infty} \|\bar{\psi}(x + \frac{p_n + q_n}{2}, T_n) - \psi(x, t_n')\|_{C^0(\mathbf{R})} = 0$$

holds. By Theorem 1.1 (2), if $n \in \{1, 2, \cdots\}$ is sufficiently large, then there exist x_n and $t_n \in \mathbf{R}$ such that

$$\begin{aligned}&\|\bar{\psi}(x, t+T_n) - \psi(x - x_n, t + T_n - t_n)\|_{C^0(\mathbf{R})}\\ &\le Ce^{-\gamma t}\|\bar{\psi}(x + \frac{p_n + q_n}{2}, T_n) - \psi(x, t'_n)\|_{C^0(\mathbf{R})}\end{aligned}$$

holds for all $t \ge 0$. Therefore, we obtain

$$\lim_{n\to\infty} \sup_{t \ge T_n - t_n} \|\bar{\psi}(x + x_n, t + t_n) - \psi(x, t)\|_{C^0(\mathbf{R})} = 0. \tag{1.4}$$

Hence, from (1.3),

$$\lim_{n\to\infty} \|\psi(x, T_n - t_n) - (\phi(x - (p_n - x_n)) + \phi(-x + (q_n - x_n)))\|_{C^0(\mathbf{R})} = 0$$

holds. Because $\lim_{n\to\infty}((p_n - x_n) - (q_n - x_n)) = +\infty$ also holds, by Theorem 1.1 (1), we obtain $\lim_{n\to\infty}(T_n - t_n) = -\infty$.

Now, we show that there exists $\bar{t}_0 \in \mathbf{R}$ such that $\lim_{n\to\infty} t_n = \bar{t}_0$ holds. Assume that there exist $\{N_n\}_{n=1}^{\infty}$ and $\{M_n\}_{n=1}^{\infty} \subset \{1, 2, \cdots\}$ such that $\lim_{n\to\infty} N_n = \lim_{n\to\infty} M_n = \infty$ and $\inf_{n=1,2,\cdots}(t_{N_n} - t_{M_n}) > 0$ hold. Then, by (1.4),

$$\lim_{n\to\infty} \|\psi(x, t) - \psi(x + x_{N_n} - x_{M_n}, t + t_{N_n} - t_{M_n})\|_{C^0(\mathbf{R})} = 0$$

holds for all $t \in \mathbf{R}$. This is contradiction with $\inf_{n=1,2,\cdots}(t_{N_n} - t_{M_n}) > 0$. Hence, $\lim_{n\to\infty} t_n = \bar{t}_0 \in \mathbf{R}$ holds.

Because $\lim_{n\to\infty}(T_n - t_n) = -\infty$ and $\lim_{n\to\infty} t_n = \bar{t}_0 \in \mathbf{R}$ hold, we obtain $T = \lim_{n\to\infty} T_n = -\infty$. Also, by (1.4),

$$\lim_{(n,m)\to(\infty,\infty)} \|\psi(x, t - \bar{t}_0) - \psi(x + x_n - x_m, t - \bar{t}_0)\|_{C^0(\mathbf{R})} = 0$$

holds for all $t \in \mathbf{R}$. Hence, we have $\lim_{(n,m)\to(\infty,\infty)} |x_n - x_m| = 0$. There exists $\bar{x}_0 \in \mathbf{R}$ such that $\lim_{n\to\infty} x_n = \bar{x}_0$ holds. Therefore, by (1.4), we obtain $\bar{\psi}(x + \bar{x}_0, t + \bar{t}_0) = \psi(x, t)$. q.e.d.

Definition 1 *For $l > 0$, $\delta \in (0, \min\{\alpha, 1 - \alpha\})$ and $L > 0$, a closed subset $\Xi_{l,\delta,L}$ of $BU(\mathbf{R})$ is defined by*

$$\begin{aligned}\Xi_{l,\delta,L} = \{u \in BU(\mathbf{R})\,|\, &0 \le u(x) \le \alpha - \delta\ (|x| < l - L),\\ 0 \le u(x) \le 1\ (l - L \le |x| \le l + L),\ &\alpha + \delta \le u(x) \le 1\ (l + L < |x|)\}.\end{aligned}$$

For $\bar{l} > 0$, $\bar{\delta} \in (0, \min\{\alpha, 1-\alpha\})$ *and* $\bar{L} > 0$, *a closed subset* $\Pi_{\bar{l},\bar{\delta},\bar{L}}$ *of* $BU(\mathbf{R})$ *is defined by*

$$\Pi_{\bar{l},\bar{\delta},\bar{L}} = \bigcup_{l \geq \bar{l}} \Xi_{l,\bar{\delta},\bar{L}}.$$

The following proposition is proved in Section 6.

Proposition 1.3 *For any* $\bar{\delta}_0 \in (0, \min\{\alpha, 1-\alpha\})$, $\bar{L}_0 > 0$ *and* $\varepsilon > 0$, *there exist* $\bar{l}_0 > 0$, $L > 0$ *and* $T > 0$ *such that for any* $l \geq \bar{l}_0$ *and* $u_0 \in \Xi_{l,\bar{\delta}_0,\bar{L}_0}$, *there exist* x_1, $x_2 \in [l-L, l+L]$ *and a solution* $u \in C([0,+\infty), BU(\mathbf{R}))$ *of* $u_t = u_{xx} + f(u)$ *with* $u(0) = u_0$ *such that*

$$\|u(x,T) - (\phi(x - x_1 - cT) + \phi(-x - x_2 - cT))\|_{C^0(\mathbf{R})} < \varepsilon$$

holds.

Theorem 1.1 and Proposition 1.3 lead to the following.

Corollary 1.4 *For any* $\bar{\delta}_0 \in (0, \min\{\alpha, 1-\alpha\})$, $\bar{L}_0 > 0$, $T_0 \in \mathbf{R}$ *and* $\varepsilon > 0$, *there exists* $\bar{l}_0 > 0$ *such that for any* $u_0 \in \Pi_{\bar{l}_0,\bar{\delta}_0,\bar{L}_0}$, *there exist* $x_0 \in \mathbf{R}$, $t_0 \geq -T_0$ *and a solution* $u \in C([0,+\infty), BU(\mathbf{R}))$ *of* $u_t = u_{xx} + f(u)$ *with* $u(0) = u_0$ *such that*

$$\sup_{t \geq T_0} \|u(x + x_0, t + t_0) - \psi(x,t)\|_{C^0(\mathbf{R})} < \varepsilon$$

holds.

Proof. We first show that there exist $M > 0$ and $\varepsilon' \in (0, \varepsilon)$ such that for any p, q and $t \in \mathbf{R}$, if

$$p + q \geq M$$

and

$$\|\psi(x,t) - (\phi(x-p) + \phi(-x-q))\|_{C^0(\mathbf{R})} < \left(1 + \frac{1}{2C}\right)\varepsilon' \tag{1.5}$$

hold, then $t \leq T_0$ holds. Assume that there exist $\{p_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty} \subset \mathbf{R}$ and $\{t_n\}_{n=1}^{\infty} \subset (T_0, +\infty)$ such that

$$\lim_{n \to \infty} (p_n + q_n) = +\infty$$

$$\lim_{n\to\infty} \|\psi(x,t_n) - (\phi(x-p_n) + \phi(-x-q_n))\|_{C^0(\mathbf{R})} = 0$$

hold. Then, from Corollary 1.2, $T_0 = -\infty$ holds. This is contradiction for $T_0 \in \mathbf{R}$.

By Proposition 1.3, there exist L, T and $\bar{l}_0' > 0$ such that for any $l \geq \bar{l}_0'$ and $u_0 \in \Xi_{l,\bar{\delta}_0,\bar{L}_0}$, there exist x_1 and $x_2 \geq l - (L - cT)$ such that

$$\|u(x,T) - (\phi(x-x_1) + \phi(-x-x_2))\|_{C^0(\mathbf{R})} < \min\left\{\frac{\varepsilon'}{2C}, \frac{\delta}{2}\right\} \tag{1.6}$$

holds. Then, let $\bar{l}_0 > 0$ be sufficiently large. Because $\frac{x_1+x_2}{2} > 0$ is sufficiently large, by Theorem 1.1 (1), there exists $t_0' \in \mathbf{R}$ such that

$$\|\psi(x,t_0') - (\phi(x - \frac{x_1+x_2}{2}) + \phi(-x - \frac{x_1+x_2}{2}))\|_{C^0(\mathbf{R})}$$
$$< \min\left\{\frac{\varepsilon'}{2C}, \frac{\delta}{2}\right\}$$

holds. Therefore, we have

$$\|u(x + \frac{x_1-x_2}{2}, T) - \psi(x,t_0')\|_{C^0(\mathbf{R})} < \min\{\varepsilon'/C, \delta\}.$$

Hence, by Theorem 1.1 (2), there exist x_0 and $t_0 \in \mathbf{R}$ such that

$$\sup_{t\geq T} \|u(x,t) - \psi(x-x_0, t-t_0)\|_{C^0(\mathbf{R})} < \varepsilon' \tag{1.7}$$

holds. Hence, from (1.6), we have

$$\|\psi(x, T-t_0) - (\phi(x-(x_1-x_0)) + \phi(-x-(x_2+x_0)))\|_{C^0(\mathbf{R})}$$
$$< \left(1 + \frac{1}{2C}\right)\varepsilon'.$$

Because $(x_1 - x_0) + (x_2 + x_0)$ is sufficiently large and (1.5) holds, $T - t_0 \leq T_0$ holds. Hence, from (1.7), $\sup_{t\geq T_0} \|u(x+x_0, t+t_0) - \psi(x,t)\|_{C^0(\mathbf{R})} < \varepsilon$ holds.

In order to prove Theorem 1.1, we need to construct a *global* invariant manifold with asymptotic stability. Here, the word of *global* means that the invariant manifold includes a solution having two interfaces at any sufficiently large distance. In Section 2, we construct a semilinear prabolic system. The system concludes a part of the reaction-diffusion equation. This is the part which consists of solutions near pairs of the travelling wave solutions at a large distance. Further, such pairs are contained in a two-dimensional *linear* subspace of the system. Hence, we can construct a global invariant manifold near the subspace by a standard technique. While we do it in Section 5, we state the result in the end of Section 2. In Section 3, we prove that there is a solution in the invariant manifold of the system and the solution satisfies Theorem 1.1 (1) in the reaction-diffusion equation, i.e., it becomes the pair of the travelling wave solutions as $t \to -\infty$. This solution is denoted by $\psi(x,t)$. In Section 4, we show that the set of solutions $\psi(x-x_0, t-t_0)$ by translation of $\psi(x,t)$ coresponds the invariant manifold of the system. This argument is rather troublesome. Then, we show Theorem 1.1 (2), i.e., the set has asymptotic stability in the reaction-diffusion equation. This is also a little troblesome, as the topologies of the equation and the system are different. Proposition 1.3 is proved in Section 6.

Acknowledgments I wish to thank Professor Hisashi Okamoto. He gave encouragement to me. I also thank Professor Hirokazu Ninomiya for helpful discussion.

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